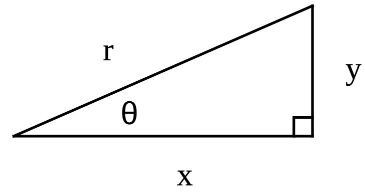
Trigonometric Identities

Basic Identities:

$$\sin\theta = \frac{y}{r} \qquad (1)$$



$$\cos\theta = \frac{x}{r} \qquad (2)$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \qquad (3)$$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \qquad (4)$$

$$\sec\theta = \frac{r}{x} = \frac{1}{\cos\theta} \qquad (5)$$

$$\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta} \qquad (6)$$

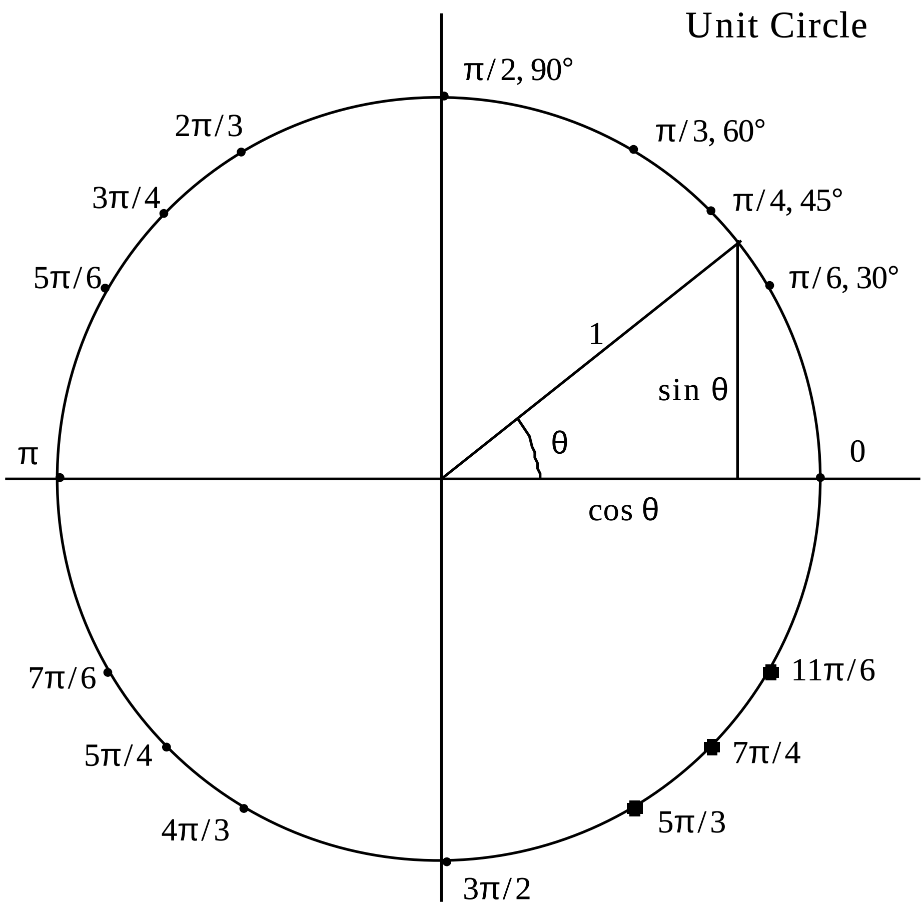


Table of Values from 0 to 2π:

degrees	0	30	45	60	90	120	135	150	180
radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin θ (y/r)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos θ (x/r)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan θ (y/x)	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\to \pm\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
cot θ (x/y)	$\to \pm\infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\to \pm\infty$
sec θ (r/x)	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\to \pm\infty$	-2	$-\sqrt{2}$	$\frac{-2\sqrt{3}}{3}$	-1
csc θ (r/y)	$\to \pm\infty$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\to \pm\infty$

Table continued

degrees	210	225	240	270	300	315	330	360
radians	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
sin θ (y/r)	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos θ (x/r)	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ (y/x)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\to \pm\infty$	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0
cot θ (x/y)	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\frac{-\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\to \pm\infty$
sec θ (r/x)	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	$\to \pm\infty$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
csc θ (r/y)	-2	$-\sqrt{2}$	$\frac{-2\sqrt{3}}{3}$	-1	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	$\to \pm\infty$

Some useful trigonometric relationships:

$$\sin^2\theta + \cos^2\theta = 1 \qquad (7)$$

$$1 + \tan^2\theta = \sec^2\theta \quad \text{(divide [7] by } \sin^2\theta) \qquad (8)$$

$$1 + \cot^2\theta = \csc^2\theta \quad \text{(divide [7] by } \cos^2\theta) \qquad (9)$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad (10)$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad (11)$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \qquad (12)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \qquad (13)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \qquad (14)$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad (15)$$

$$\cos 2q = \cos^2 q - \sin^2 q = 2\cos^2 q - 1 = 1 - 2\sin^2 q \qquad (16)$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad (17)$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}} \qquad (18)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}} \qquad (19)$$

$$\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta} \qquad (20)$$

All of the above relationships are easily proved from Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad (21a)$$

and it also follows that

$$e^{-i\theta} = \cos\theta - i\sin\theta, \qquad (21b)$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(-\theta) \qquad (22)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\sin(-\theta) \qquad (23)$$

and these identities can be manipulated to get a new and sometimes more convenient expression for the trigonometric function of an angle. Just in case you doubt this method, we append some derivations:

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 \\ &= \frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4} + \frac{e^{2i\theta} - 2 + e^{-2i\theta}}{-4} \\ &= \frac{e^{2i\theta} + 2 + e^{-2i\theta} - e^{2i\theta} + 2 - e^{-2i\theta}}{4} \\ &= 1\,.\end{aligned} \qquad (24)$$

$$\sin 2\theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}$$

$$= \frac{\left(e^{i\theta}\right)^2 - \left(e^{-i\theta}\right)^2}{2i}$$

$$= \frac{(e^{i\theta} + e^{-i\theta})(e^{i\theta} - e^{-i\theta})}{2i} \qquad (25)$$

$$= (e^{i\theta} + e^{-i\theta}) \sin\theta$$

$$= 2\cos\theta \sin\theta \,.$$

The hyperbolic functions are analogous to the trig functions and often arise in physical situations. Their relations to the trig functions are as follows:

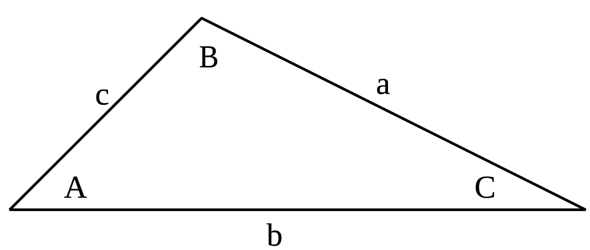
$$\sinh x = \frac{e^{x} - e^{-x}}{2} = \frac{e^{-i(ix)} - e^{i(ix)}}{2} = -i \sin(ix)\,, \qquad (26)$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} = \frac{e^{-i(ix)} + e^{i(ix)}}{2} = \cos(ix)\,, \qquad (27)$$

$$\cos x = \cosh\frac{x}{i} = \cosh(-ix) = \cosh(ix)\,, \qquad (28)$$

$$\sin x = i \sinh\frac{x}{i} = i \sinh(-ix) = -i \sinh(ix). \qquad (29)$$

(The following laws apply for all triangles with angles, A, B and C and opposite side lengths as defined in the figure.)



Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (30)

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$ (31)

Derivatives of Trig Functions:

$$\frac{d}{dx}(\sin u) = \cos u\left(\frac{du}{dx}\right) \quad (32)$$

$$\frac{d}{dx}(\cos u) = -\sin u\left(\frac{du}{dx}\right) \quad (33)$$

$$\frac{d}{dx}(\tan u) = \sec^2 u\left(\frac{du}{dx}\right) \quad (34)$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u\left(\frac{du}{dx}\right) \quad (35)$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u\left(\frac{du}{dx}\right) \quad (36)$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u\left(\frac{du}{dx}\right) \quad (37)$$

References: Shenk, Calculus and Analytic Geometry

Boas, Mathematical Methods in the Physical Sciences